

Approximate Symmetry Analysis of a Class of Perturbed Nonlinear Reaction-Diffusion Equations

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Abstract

In this paper, the problem of approximate symmetries of a class of nonlinear reaction-diffusion equations called Kolmogorov-Petrovsky-Piskounov (KPP) equation is comprehensively analyzed. In order to compute the approximate symmetries, we have applied the method which was proposed by Fushchich and Shtelen [8] and fundamentally based on the expansion of the dependent variables in a perturbation series. Particularly, an optimal system of one dimensional subalgebras is constructed and some invariant solutions corresponding to the resulted symmetries are obtained.

Keywords: Approximate symmetry, Approximate solution, Lie group analysis, Kolmogorov-Petrovsky-Piskounov (KPP) equation.

1 Introduction

Nonlinear problems arise widely in various fields of science and engineering mainly due to the fact that most physical systems are inherently nonlinear in nature. But for nonlinear partial differential equations (PDEs), analytical solutions are rare and difficult to obtain. Hence, the investigation of the exact solutions of nonlinear PDEs plays a fundamental role in the analysis of nonlinear physical phenomena. One of the most famous and established procedures for obtaining exact solutions of differential equations is the classical symmetries method, also called group analysis. This method was originated in 1881 from the pioneering work of Sophus Lie [12]. The investigation of symmetries has been manifested as one of the most significant and fundamental methods in almost every branch of science such as in mathematics and physics. Nowadays,

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the application of Lie group theory for the construction of solutions of nonlinear PDEs can be regarded as one of the most active fields of research in the theory of nonlinear PDEs and many good books have been dedicated to this subject (such as [4, 14, 15]). For some nonlinear problems, however symmetries are not rich to determine useful solutions. Hence, this fact was the motivation for the creation of several generalizations of the classical Lie group method. Consequently, several alternative reduction methods have been introduced, going beyond Lie's classical procedure and providing further solutions. One of the techniques widely applied in analyzing nonlinear problems is the perturbation analysis. Perturbation theory comprises mathematical methods that are applied to obtain an approximate solution to a problem which can not be solved exactly. Indeed, this procedure is performed by expanding the dependent variables asymptotically in terms of a small parameter. In order to combine the power of the Lie group theory and perturbation analysis, two different approximate symmetry theories have been developed recently. The first method is due to Baikov, Gazizov and Ibragimov [2, 3]. Successively another method for obtaining approximate symmetries was introduced by Fushchich and Shtelen [8].

In the method proposed by Baikov, Gazizov and Ibragimov, the Lie operator is expanded in a perturbation series other than perturbation for dependent variables as in the usual case. In other words, assume that the perturbed differential equation be in the form: $F(z) = F_0(z) + \varepsilon F_1(z)$, where $z = (x, u, u_{(1)}, \dots, u_{(n)})$, F_0 is the unperturbed equation, $F_1(z)$ is the perturbed term and $X = X^0 + \varepsilon X^1$ is the corresponding infinitesimal generator. The exact symmetry of the unperturbed equation $F_0(z)$ is denoted by X^0 and can be obtained as $X^0 F_0(z) \big|_{F_0(z)=0} = 0$. Then, by applying the auxiliary function $H = \frac{1}{\varepsilon} X^0 (F_0(z) + \varepsilon F_1(z)) \big|_{F_0+\varepsilon F_1=0}$, vector field X_1 will be deduced from the following relation:

$$X^1 F_0(z) \big|_{F_0=0} + H = 0. \quad (1)$$

Finally, after obtaining the approximate symmetries, the corresponding approximate solutions will be obtained via the classical Lie symmetry method [10]. In the second method due to Fushchich and Shtelen, first of all the dependent variables are expanded in a perturbation series. In the next step, terms are then separated at each order of approximation and as a consequence a system of equations to be solved in a hierarchy is determined. Finally, the approximate symmetries of the original equation is defined to be the exact symmetries of the system of equations resulted from perturbations [6, 8, 18]. Pakdemirli et al. in a recent paper [16] have compared these above two methods. According to their comparison, the expansion of the approximate operator applied in the first method, does not reflect well an approximation in the perturbation sense; While the second method is consistent with the perturbation theory and results correct terms for the approximate solutions. Consequently, the second method is superior to the first one according to the comparison in [16].

Nonlinear reaction-diffusion equations can be regarded as mathematical models which explain the change of the concentration of one or more substances distributed in space. Indeed, this variation occurs under the influence of two main processes including chemical reactions in which the substances are locally transformed into each other, and diffusion which makes the substances to spread out over a surface in space. From the mathematical point of view, reaction-diffusion systems generally take the form of semi-linear parabolic PDEs. It is worth mentioning that the solutions of reaction-diffusion equations represent a wide range of behaviors, such as formation of wave-like phenomena and traveling waves as well as other self-organized patterns.

In this paper, we will apply the method proposed by Fushchich and Shtelen [8] in order to present a comprehensive analysis of the approximate symmetries of a significant class of nonlinear reaction-diffusion equations called Kolmogorov-Petrovsky-Piskounov (KPP) equation [11]. This equation can be regarded as the most simple reaction-diffusion equation concerning the concentration u of a single substance in one spatial dimension and is generally defined as follows:

$$u_t - u_{xx} = R(u). \quad (2)$$

By inserting different values to the reaction term $R(u)$ of equation (2), the following significant equations are deduced:

1. If the reaction term $R(u)$ vanishes, then the resulted equation displays a pure diffusion process and defined by:

$$u_t = u_{xx}, \quad (3)$$

Note that the above equation is called Fick's second law [11].

2. By inserting $R(u) = au(1 - u)$, $a \geq 0$, the Fisher equation (or logistic equation) is resulted as follows:

$$u_t = u_{xx} + au(1 - u), \quad (4)$$

This equation can be regarded as the archetypical deterministic model for the spread of a useful gene in a population of diploid individuals living in a one dimensional habitat [7, 17].

3. By inserting $R(u) = u^2(1 - u)$, the Zeldovich equation will be deduced as follows:

$$u_t = u_{xx} + u^2(1 - u), \quad (5)$$

This equation appears in combustion theory. The unknown u displays temperature, while the last term on the right-hand side is concerned to the generation of heat by combustion [5, 9].

4. By inserting $R(u) = u(1 - u^2)$ the Newell-Whitehead-Segel (NWS) equation (or amplitude equation) is resulted as follows:

$$u_t = u_{xx} + u(1 - u^2). \quad (6)$$

This equation arises in the analysis of thermal convection of a fluid heated from below after carrying out a suitable normalization [13].

This paper is organized as follows: Section 2 is devoted to the thorough investigation of the approximate symmetries and approximate solutions of the KPP equation. For this purpose, we will concentrate on the four special and significant forms of the KPP equation described above i.e Fick's second law, Fisher's equation, Zeldovich equation and Newell-Whitehead-Segel (NWS) equation. In section 3, an optimal system of subalgebras is constructed and the corresponding symmetry transformations are obtained. Some concluding remarks are mentioned at the end of the paper.

2 Approximate symmetries of the KPP equation

In this section, first of all the problem of exact and approximate symmetries of the Fick's second law (3) with a small parameter is investigated. Then the approximate symmetries and the exact and approximate invariant solutions corresponding to the perturbed Fisher's equation, Zeldovich equation and Newell-Whitehead-Segel (NWS) equation will be determined.

2.1 Exact symmetries of the perturbed Fick's second law

The perturbed Fick's second law is defined as follows:

$$u_t = \varepsilon u_{xx}. \quad (7)$$

where ε is a small parameter. Let X be the infinitesimal symmetry generator corresponding to the equation (7) which is defined as follows:

$$X = \xi(x, t, u)\partial_x + \tau(x, t, u)\partial_t + \varphi(x, t, u)\partial_u. \quad (8)$$

Now by acting the second prolongation of the symmetry operator (8) on equation (7), an overdetermined system of equations for ξ, τ and φ will be obtained. By solving this resulted determining equations, it is inferred that:

$$\begin{aligned} \xi &= (c_1 t x + c_2 x) - 2\varepsilon c_4 t + c_6, \\ \tau &= c_1 t^2 + 2c_2 t + c_3, \\ \varphi &= (c_4 x + c_5 - c_1 t/2 - c_1 x^2/4\varepsilon)u + F(x, t). \end{aligned} \quad (9)$$

where $F(x, t)$ is an arbitrary function satisfying the perturbed Fick's second law equation (8), and c_i , $i = 1, \dots, 6$ are arbitrary constants. Hence, this equation admits a six-dimensional Lie algebra with the following generators:

$$\begin{aligned} X_1 &= \partial_x, & X_4 &= -2\varepsilon t \partial_x + xu \partial_u, \\ X_2 &= \partial_t, & X_5 &= u \partial_u, \\ X_3 &= x \partial_x + 2t \partial_t, & X_6 &= 4xt \partial_x + 4t^2 \partial_t - (2t + x^2/\varepsilon)u \partial_u. \end{aligned} \quad (10)$$

plus the following infinite dimensional subalgebra which is spanned by $X_F = F(x, t) \partial_u$, where F satisfies (8).

2.2 Exact invariant solutions

In this part, we compute some exact invariant solutions corresponding to the resulted infinitesimal generators.

Case 1. Consider the symmetry operator $X = cX_1 + X_2$, where c is a constant.

Now taking into account [4, 14, 15], by applying the Lie symmetry reduction technique the corresponding exact and approximate invariant solutions will be obtained as follows. The characteristic equation associated to the symmetry generator X is given by $dx/c = dt/1 = du/0$. By solving above equation, the following Lie invariants are resulted: $x - ct = y$, $u = v(y)$. By substituting these invariants into equation (7) we obtain: $\varepsilon v''(y) + cv'(y) = 0$. Consequently, by solving the above resulted ODE, the following solution is deduced for equation (7): $u(x, t) = c_1 + c_2 \exp(-c(x - ct)/\varepsilon)$.

Case 2. For the symmetry generator X_3 , the corresponding characteristic equation is: $dx/x = dt/2t = du/0$. Thus, these Lie invariants are determined: $u = v(y)$, $y = x^2/t$. By substituting above invariants into (7) the following ODE is inferred: $4\varepsilon y v''(y) + v'(y)(2\varepsilon + y) = 0$. Hence, another solution is deduced for equation (7): $u = v(y) = c_1 + c_2 \operatorname{erf}(|x|/2\sqrt{\varepsilon t})$, where c_1 and c_2 are arbitrary constants, and erf is the error function given by: $\operatorname{erf}(x) = (2/\sqrt{\pi}) \int_0^x e^{-t^2} dt$.

2.3 Perturbed Fisher's equation

In this section, a thorough investigation of the symmetries of the perturbed Fisher's equation is proposed:

$$u_t = \varepsilon u_{xx} + au(1 - u) \quad (11)$$

For this purpose, firstly the exact symmetries of the perturbed Fisher's equation (11) will be calculated. Then, the approximate symmetries of this equation will be analyzed.

Now by acting the second prolongation of the symmetry generator (8) on the perturbed Fisher's equation and solving the resulted determining equations, it is deduced that $\xi = c_2$, $\tau = c_1$, and $\varphi = 0$. where c_1 and c_2 are arbitrary constants. Hence, the following exact trivial symmetries are obtained: $X_1 = \partial_x$, $X_2 = \partial_t$. For the infinitesimal symmetry generator $X = c\partial_x + \partial_t$, the corresponding characteristic equation is given by $dx/c = dt/1 = du/0$.

Therefore, the Lie invariants are resulted as $x - ct = y$ and $u = v(y)$. After substituting these invariants into the perturbed Fisher's equation, the following reduced ordinary differential equation is obtained:

$$\varepsilon v''(y) + cv'(y) + av(y)(1 - v(y)) = 0. \quad (12)$$

But it is worth noting that finding an exact solution for the differential equation (12) is difficult. For the particular case $c = \pm 5/\sqrt{6}$, Ablowitz and Zepptella [1] used Painleve's singularity structure analysis in order to obtain the first corresponding explicit analytical solution which is given by:

$$v(y) = u(x, t) = \left[1 + \frac{\varepsilon}{\sqrt{6}} \exp\left(\sqrt{6}x - \frac{5}{6}t\right) \right]^{-2}. \quad (13)$$

2.3.1 Approximate symmetries of the perturbed Fisher's equation

In this section, we apply the method proposed in [8] in order to analyze the problem of approximate symmetries of the Fisher's equation with an accuracy of order one. First, we expand the dependent variable in perturbation series, and then we separate terms of each order of approximation, so that a system of equations will be formed. The derived system is assumed to be coupled and its exact symmetry will be considered as the approximate symmetry of the original equation.

We expand the dependant variable up to order one as follows:

$$u = v + \varepsilon w, \quad 0 < \varepsilon \leq 1. \quad (14)$$

Where v and w are smooth functions of x and t . After substitution of (14) into the perturbed Fisher's equation (11) and equating to zero the coefficients of $o(\varepsilon^0)$ and $o(\varepsilon^1)$, the following system of partial differential equations is resulted:

$$O(\varepsilon^0) : v_t - av(1 - v)w = 0, \quad O(\varepsilon) : w_t - v_{xx} - aw(1 - 2v) = 0. \quad (15)$$

Definition: The approximate symmetry of the Fisher's equation with a small parameter is called the exact symmetry of the system of differential equations (15).

Now, consider the following symmetry transformation group acting on the PDE system (15):

$$\begin{aligned} \tilde{x} &= x + a\xi_1(t, x, v, w) + o(a^2), & \tilde{t} &= t + a\xi_2(t, x, v, w) + o(a^2), \\ \tilde{v} &= v + a\varphi_1(t, x, v, w) + o(a^2), & \tilde{w} &= w + a\varphi_2(t, x, v, w) + o(a^2), \end{aligned} \quad (16)$$

where a is the group parameter and ξ_1, ξ_2 and φ_1, φ_2 are the infinitesimals of the transformations for the independent and dependent variables, respectively. The associated vector field is of the form:

$$X = \xi_1(t, x, v, w)\partial_t + \xi_2(t, x, v, w)\partial_x + \varphi_1(t, x, v, w)\partial_v + \varphi_2(t, x, v, w)\partial_w. \quad (17)$$

The invariance of the system (15) under the infinitesimal symmetry transformation group (17) leads to the following invariance condition: $pr^{(2)}X[\Delta] = 0$, and $\Delta = 0$. Hence, the following set of determining equations is inferred:

$$\partial_w \xi_2 = 0, \quad av^2 \partial_w \xi_1 + \partial_w \varphi_1 - av \partial_w \xi_1 = 0, \quad \dots, \quad 2\partial_{vx} \xi_2 - \partial_{vv} \varphi_1 = 0. \quad (18)$$

By solving this system of PDEs, it is deduced to: $\xi_2 = C_1 x + C_3$, $\varphi_1 = 0$, $\xi_1 = C_2$, $\varphi_2 = -2C_1 w$, where C_1, C_2 and C_3 are arbitrary constants. Thus, the Lie algebra of the resulted infinitesimal symmetries of the PDE system (15) is spanned by these three vector fields:

$$X_1 = \partial_t, \quad X_2 = \partial_x, \quad X_3 = x\partial_x - 2w\partial_w. \quad (19)$$

2.3.2 Approximate invariant solutions

In this section, the approximate solutions will be obtained from the approximate symmetries which were resulted in the pervious section.

Case 1. $X = x\partial_x - 2w\partial_w$. By applying the classical Lie symmetry group method, the corresponding characteristic equation is $dx/x = dt/0 = dv/0 = dw/(-2w)$. So that the resulted invariants are: $t = T$, $v = f(T)$ and $w = g(T)/x^2$. After substituting these invariants into the first equation of the PDE system (15), we have:

$$f'(T) - af(T)(1 - f(T)) = 0. \quad (20)$$

Consequently, the following solution is obtained:

$$f(T) = v = 1/(1 + c_1 e^{-at}). \quad (21)$$

After substituting v in the second equation of the PDE system (15), this ODE is resulted $g'(T) + ag(T)[2/(1 + c_1 e^{-at}) - 1] = 0$. Therefore, we have $g(T) = c_2 e^{-at}/(1 + c_1 e^{-at})^2$. Finally, taking into account (14), the following approximate solution is inferred:

$$u(x, t) = v + \varepsilon w = 1/(1 + c_1 e^{-at}) + \varepsilon c_2 e^{-at}/(x^2 (1 + c_1 e^{-at})^2), \quad (22)$$

where c_1 and c_2 are arbitrary constants.

Case 2. Now consider $X = X_1 + cX_2$ where c is an arbitrary constant. The corresponding characteristic equation is defined by $dx/c = dt/1 = dv/0 = dw/0$. So, the associated Lie invariants are $x - ct = y$, $v = f(y)$, and $w = g(y)$. By substituting the resulted invariants into the first equation of the PDE system (15), the reduced equation is determined as $cf'(y) + af(y)(1 - f(y)) = 0$. Therefore, we have $v(x, t) = 1/(c_1 e^{a(x-ct)/c} + 1)$. Now by substituting $v(x, t)$ into the second equation of the PDE system (15), it is inferred that:

$$cg(y) + \frac{c_1 a^2 e^{ay/c} (-1 + c_1 e^{ay/c})}{c_2 (1 + c_1 e^{ay/c})^3} + g(y) \left(1 - \frac{2}{c_1 e^{ay/c} + 1} \right) = 0. \quad (23)$$

By solving above equation, we have:

$$g(y) = \frac{e^{ay/c}}{(c_1 e^{ay/c} + 1)^2} \left(c_1 \frac{a^2}{c_3} y - \frac{2ac_1}{c_2} \ln(c_1 e^{ay/c} + 1) + c_2 \right). \quad (24)$$

Finally, the following approximate solution is resulted:

$$u(x, t) = v + \varepsilon w = \frac{1}{c_1 e^{a(x-ct)/c} + 1} \left\{ 1 + \varepsilon e^{a(x-ct)/c} \left(\frac{c_1}{c^3} (x - ct) - \frac{2ac_1}{c^2} \ln(c_1 e^{a(x-ct)/c} + 1) + c_2 \right) \right\}. \quad (25)$$

Consequently, the approximate solutions corresponding to all the resulted operators were computed.

2.4 Perturbed Zeldovich equation

In this section, we will investigate the exact and approximate symmetries of the Zeldovich equation with a small parameter:

$$u_t - \varepsilon u_{xx} = u^2(1 - u). \quad (26)$$

For this purpose, first of all we will compute the exact symmetries, then by applying the classical Lie symmetry method, the perturbed Zeldovich equation would be converted to an ODE.

By acting the symmetry operator (8) on the perturbed Zeldovich equation (26) and solving the resulted determining equations we have: $\xi = c_1$, $\tau = c_2$, $\varphi = 0$, where c_1 and c_2 are arbitrary constants. Hence, the corresponding infinitesimal symmetries will be spanned by these two vector fields $X_1 = \partial_t$, and $X_2 = \partial_x$. The characteristic equation corresponding to the symmetry operator $X = X_1 + cX_2$ is given by $dx/c = dt/1 = du/0$. Hence, the Lie invariants are obtained as $x - ct = y$, and $u = f(y)$. After substituting these invariants into equation (26), the reduced equation is inferred as $\varepsilon f''(y) + cf'^2(y)(1 - f(y)) = 0$.

2.4.1 Approximate Symmetries of the Zeldovich equation

In this section, we use the method proposed in [8] in order to obtain the approximate symmetries of the equation (26) with the accuracy $o(\varepsilon)$. By expanding the dependent variable of this equation in perturbation series we have:

$$u = v + \varepsilon w, \quad 0 \leq \varepsilon \leq 1. \quad (27)$$

Then by substituting the above relation into the perturbed equation (26) and separating terms of each order of approximation, the following equations with respect to $o(\varepsilon^0)$ and $o(\varepsilon^1)$ are deduced:

$$O(\varepsilon^0) : v_t - v^2(1 - v) = 0, \quad O(\varepsilon^1) : w_t - v_{xx} - 2vw(1 - v) + v^2w = 0. \quad (28)$$

It is worth mentioning that the resulted approximate symmetries of the differential equation (26) correspond to the exact symmetries of the PDE system (28).

Now, by acting the second prolongation of the infinitesimal symmetry operator (17) on the PDE system (28) and solving the resulted determining equations, we have $\xi_1 = c_2$, $\xi_2 = c_1x + c_3$, $\varphi_1 = 0$, and $\varphi_2 = -2c_1w$, where c_1, c_2 and c_3 are arbitrary constants. Consequently, the Lie algebra of the symmetry generators corresponding to the PDE system (28) is spanned by:

$$X_1 = \partial_t, \quad X_2 = \partial_x, \quad X_3 = x\partial_x - 2w\partial_w. \quad (29)$$

2.4.2 Approximate invariant solutions

Now, we obtain the approximate invariant solutions corresponding to the perturbed equation (26). For the symmetry operator X_3 the corresponding characteristic equation is given by $dx/x = dt/0 = dv/0 = dw/(-2w)$. So, the invariants are resulted as $t = T$, $v = f(T)$, and $w = g(T)/x^2$. By inserting these invariants into the first equation of the PDE system (28), the reduced equation is $f'^2(T)(1 - f(T)) = 0$. Therefore, we have $v = f(T) = 1/W(-e^{-t-1}/c_1)$, where the function $W(z)$ is defined implicitly by this equation $z = W(z)e^{W(z)}$. After substituting this resulted solution into the second equation of the PDE system (28), we obtain $g'^2 = 0$. The solution of the above equation is:

$$g(T) = \frac{c_2 \exp(-2W(-e^{-t-1}/c_1))W(-e^{-t-1}/c_1)}{W(-e^{-t-1}/c_1) + 1}. \quad (30)$$

Finally, the following approximate invariant solution for the equation (26) is deduced:

$$u(x, t) = f(T) + \varepsilon \frac{c_2 \exp(-2W(-e^{-t-1}/c_1))W(-e^{-t-1}/c_1)}{x^2 (W(-e^{-t-1}/c_1) + 1)}. \quad (31)$$

2.5 Perturbed NSW equation

Similar to the previous sections, we will analyze the symmetries of the perturbed NSW equation:

$$u_t - \varepsilon u_{xx} = u(1 - u^2). \quad (32)$$

By applying the same calculations on this equation, the approximate symmetries are resulted as $X_1 = \partial_t$, $X_2 = \partial_x$, and $X_3 = x\partial_x - 2w\partial_w$. The Lie invariants corresponding to the symmetry operator X_3 are as $t = T$, $v = f(T)$, and $w = g(T)/x^2$. Consequently, the following approximate invariant solution is deduced:

$$u(x, t) = \frac{\pm 1}{\sqrt{1 + c_1 e^{-2t}}} + \varepsilon \frac{c_2 e^{-2t}}{(1 + c_1 e^{-2t})^{3/2}}. \quad (33)$$

3 Optimal system of the KPP equation

In this section, an optimal system of subalgebras corresponding to the resulted approximate symmetries of the KPP equation is constructed. As it was shown in the previous sections, the Lie algebra of the approximate symmetries corresponding to the Fisher's equation, Zeldovich equation and Newell-Whitehead-Segel (NSW) equation is three dimensional and spanned by the following generators:

$$X_1 = \partial_t, \quad X_2 = \partial_x, \quad X_3 = x\partial_x - 2w\partial_w. \quad (34)$$

The commutation relations corresponding to these vector fields are given in Table 1.

Table1: The commutator table of the approximate symmetries of the KPP equation.

$[X_i, X_j]$	X_1	X_2	X_3
X_1	0	0	0
X_2	0	0	X_2
X_3	0	$-X_2$	0

It is worth noting that each s -parameter subgroup corresponds to one of the group invariant solutions. Since any linear combination of the infinitesimal generators is also an infinitesimal generator, there are always infinitely many distinct symmetry subgroups for a differential equation. But it's not practical to find the list of all group invariant solutions of a system; Consequently, we need an effective and systematic means of classifying these solutions, leading to an "optimal system" of group-invariant solutions from which every other such solutions can be resulted. Let G be a Lie group and \mathfrak{g} denotes its Lie algebra. An optimal system of s -parameter subgroups is indeed a list of conjugacy inequivalent s -parameter subgroups with the property that any other subgroup is conjugate to precisely one subgroup in the list. Similarly, a list of s -parameter

subalgebras forms an optimal system if every s -parameter subalgebra of \mathfrak{g} is equivalent to a unique member of the list under some element of the adjoint representation: $\tilde{h} = \text{Ad}_g(h)$, with $g \in G$.

According to the proposition (3.7) of [14], the problem of finding an optimal system of subgroups is equivalent to that of obtaining an optimal system of subalgebras. For one-dimensional subalgebras, this classification problem is essentially the same as the problem of classifying the orbits of the adjoint representation. Since each one-dimensional subalgebra is determined by a nonzero vector in \mathfrak{g} , this problem is attacked by the naive approach of taking a general element X in \mathfrak{g} and subjecting it to various adjoint transformations so as to simplify it as much as possible. Thus we will deal with the construction of an optimal system of subalgebras of \mathfrak{g} . The adjoint action is given by the Lie series: $\text{Ad}(\exp(\varepsilon X_i, X_j)) = X_j - \varepsilon[X_i, X_j] + \varepsilon^2/2[X_i, [X_i, X_j]] - \dots$, where $[X_i, X_j]$ denotes the Lie bracket, ε is a parameter and $i, j = 1, 2, 3$ ([14]).

The adjoint representation Ad corresponding to the resulted approximate symmetries is presented in table 2 with the (i, j) -th entry indicating $\text{Ad}(\exp(\varepsilon x_i)x_j)$.

Table2: Adjoint representation of the approximate symmetries of the KPP equation.

Ad	X_1	X_2	X_3
X_1	X_1	X_2	X_3
X_2	X_1	X_2	$X_3 - \varepsilon X_2$
X_3	X_1	$e^\varepsilon X_2$	X_3

Therefore, we can state the following theorem:

Theorem 1: An optimal system of one dimensional subalgebras corresponding to the Lie algebra of approximate symmetries of the KPP equation is generated by: (i) X_1 , (ii) $\alpha X_1 + X_2$, (iii) $\beta X_1 + X_3$, where $\alpha, \beta \in \mathbf{R}$ are arbitrary constants.

Proof: Let $F_i^s : \mathfrak{g} \rightarrow \mathfrak{g}$ be a linear map defined by $X \rightarrow \text{Ad}(\exp(s_i X_i)X)$ for $i = 1, \dots, 3$. The matrices M_i^s of F_i^s with respect to the basis $\{X_1, X_2, X_3\}$ are given by:

$$M_1^s = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad M_2^s = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -s_1 & 1 \end{pmatrix}, \quad M_3^s = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{s_2} & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (35)$$

Let $X = \sum_{i=1}^3 a_i X_i$, then $F_3^s \circ F_2^s \circ F_1^s : X \mapsto a_1 X_1 + a_2 e^{s_2} X_2 + (a_3 - s_1 a_2) X_3$.

In the following, by alternative action of these matrices on a vector field X , the coefficients a_i of X will be simplified.

If $a_2 \neq 0$, then we can make the coefficients of X_3 vanish by F_1^s ; By setting $s_1 = a_3/a_2$. Scaling X if necessary, we can assume that $a_2 = 1$. So, X is

reduced to the case (ii). If $a_2 = 0$ and $a_3 \neq 0$, by scaling we insert $a_3 = 1$. So X is reduced to the case (iii). Finally, if $a_2 = a_3 = 0$, then X is reduced to the case (i). There is not any more possible cases for investigating and the proof is complete.

In order to obtain the group transformations which are generated by the resulted infinitesimal symmetry generators (34), we need to solve the following system of first order ordinary differential equations ($x_1 = x, x_2 = t, u_1 = v, u_2 = w$).

$$\begin{aligned} d\tilde{x}_j(s)/ds &= \xi_i^j(\tilde{x}(s), \tilde{t}(s), \tilde{v}(s), \tilde{w}(s)), & \tilde{x}_j(0) &= x_j, & i &= 1, 2, 3. \\ d\tilde{u}_j(s)/ds &= \varphi_i^j(\tilde{x}(s), \tilde{t}(s), \tilde{v}(s), \tilde{w}(s)), & \tilde{u}_j(0) &= u_j, & j &= 1, 2. \end{aligned} \quad (36)$$

Hence, by exponentiating the resulted infinitesimal approximate symmetries of the KPP equation, the one-parameter groups $G_i(s)$ generated by X_i for $i = 1, 2, 3$ are determined as follows:

$$\begin{aligned} G_1 &: (t, x, v, w) \mapsto (t + s, x, v, w), \\ G_2 &: (t, x, v, w) \mapsto (t, x + s, v, w), \\ G_3 &: (t, x, v, w) \mapsto (t, e^s x, v, e^{-2s} w). \end{aligned} \quad (37)$$

Consequently, we can state the following theorem:

Theorem 2: If $u = f(t, x) + \varepsilon g(t, x)$ is a solution of the KPP equation, so are the following functions:

$$\begin{aligned} G_1(s) \cdot u(t, x) &= f(t - s, x) + \varepsilon g(t - s, x), \\ G_2(s) \cdot u(t, x) &= f(t, x - s) + \varepsilon g(t, x - s), \\ G_3(s) \cdot u(t, x) &= f(t, e^{-s} x) + \varepsilon e^{-2s} g(t, e^{-s} x). \end{aligned} \quad (38)$$

Conclusion

The investigation of the exact solutions of nonlinear PDEs plays an essential role in the analysis of nonlinear phenomena. Lie symmetry method greatly simplifies many nonlinear problems. Exact solutions are nevertheless hard to investigate in general. Furthermore, many PDEs in application depend on a small parameter, hence it is of great significance and interest to obtain approximate solutions. Perturbation analysis method was thus developed and it has a significant role in nonlinear science, particularly in obtaining approximate analytical solutions for perturbed PDEs. This procedure is mainly based on the expansion of the dependent variables asymptotically in terms of a small parameter. The combination of Lie group theory and perturbation theory yields two distinct approximate symmetry methods. The first method due to Baikov et al. generalizes symmetry group generators to perturbation forms [2, 3]. The second

method proposed by Fushchich and Shtelen [8] is based on the perturbation of dependent variables in perturbation series and the approximate symmetry of the original equation is decomposed into an exact symmetry of the system resulted from the perturbation. Taking into account the comparison in [16] the second method is superior to the first one.

As it is well known, the solutions of nonlinear reaction-diffusion equations represent a wide class of behaviors, including the formation of wave-like phenomena and traveling waves as well as other self-organized patterns. In this paper we have comprehensively analyzed the approximate symmetries of a significant class of nonlinear reaction-diffusion equations called Kolmogorov-Petrovsky-Piskounov (KPP) equation. For this purpose, we have concentrated on four particular and important forms of this equation including: Fick's second law, Fisher's equation, Zeldovich equation and Newell-Whitehead-Segel (NWS) equation. It is worth mentioning that in order to calculate the approximate symmetries corresponding to these equations, we have applied the second approximate symmetry method which was proposed by Fushchich and Shtelen. Meanwhile, we have constructed an optimal system of subalgebras. Also, we have obtained the symmetry transformations and some invariant solutions corresponding to the resulted symmetries.

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